# SYSTEMS IDENTIFICATION

Ali Karimpour
Associate Professor
Ferdowsi University of Mashhad

**References are appeared in the last slide.**

**Last update: 21.11.2015 (1394/08/30)**

# Lecture 4

## System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include*:

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

# System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include:*

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

# Impulse responses

**It is well known that a linear, time-invariant, causal system can be described as:**

$$y(t)=\int_{\tau=0}^{\infty} g(\tau)u(t-\tau)d\tau$$

**Sampling**

$$y(kT)=\int_{\tau=0}^{\infty} g(\tau)u(kT-\tau)d\tau$$

**Most often, the input signal *u*(*t*) is kept constant between the sampling instants:**

$$u(t)=u_k \qquad kT\le t\le (k+1)T$$

**So**

$$y(kT)=\int_{\tau=0}^{\infty} g(\tau)u(kT-\tau)d\tau=\sum_{l=1}^{\infty}\int_{\tau=(l-1)T}^{lT} g(\tau)u(kT-\tau)d\tau$$

$$=\sum_{l=1}^{\infty}\left[\int_{\tau=(l-1)T}^{lT} g(\tau)d\tau\right]u_{k-l}=\sum_{l=1}^{\infty} g_T(l)\,u_{k-l}$$

$$y(t)=\sum_{k=1}^{\infty} g(k)\,u(t-k) \qquad t=0,1,2,3.....$$

# Transfer functions

Define backward shift operator $q^{-1}$ as

$$q^{-1}u(t) = u(t-1)$$

Now we can write output as:

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) = \sum_{k=1}^{\infty} g(k)\left(q^{-k}u(t)\right) = \left[\sum_{k=1}^{\infty} g(k)q^{-k}\right]u(t) = G(q)u(t)$$

$G(q)$ is the transfer operator or transfer function

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}$$

Similarly for disturbance we have

$$v(t) = H(q)e(t)$$

So the basic description for a linear system with additive disturbance is:

$$\boxed{y(t) = G(q)u(t) + H(q)e(t)}$$

# Transfer functions

## Some terminology

$G(q)$ is the transfer operator or transfer function

$$G(q)=\sum_{k=1}^{\infty} g(k)q^{-k} \qquad \text{or} \qquad G(z)=\sum_{k=1}^{\infty} g(k)z^{-k}$$

We shall say that the transfer function $G(q)$ is stable if

$$\sum_{k=1}^{\infty} \left|g(k)\right| < \infty$$

This means that $G(z)$ is analytic on and outside the unit circle.

We shall say the filter $H(q)$ is monic if $h(0)=1$:

$$H(q)=\sum_{k=0}^{\infty} h(k)q^{-k}$$

# System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include:*

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

# Passing a random process through an LTI system

Introduction:


$$y(t)=x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau$$ continuous system

$$y_t=x_t*h_t=h_t*x_t=\sum_{k=-\infty}^{+\infty} h_k \,.\, x_{t-k}$$ discrete system

**Impulse Response** ⟶ **Transfer function**

For continuous systems

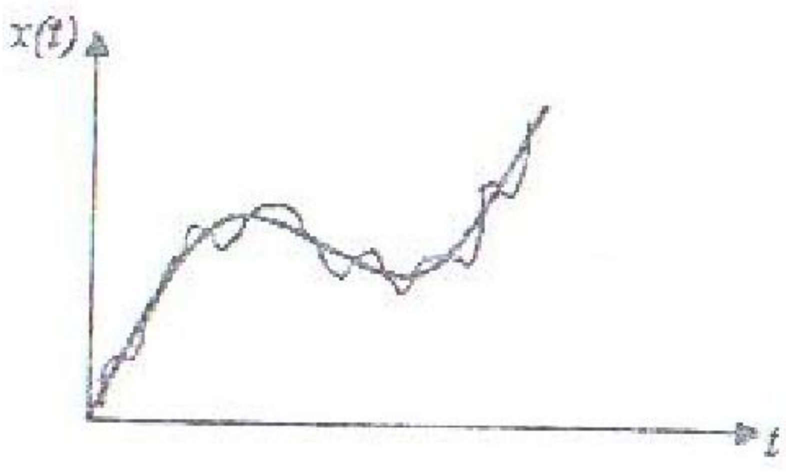
$$Y(s)=G(s).X(s)$$

For discrete systems

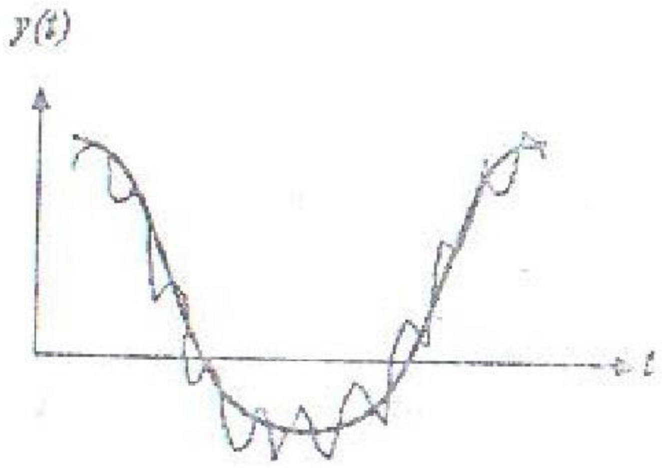
$$Y(z)=G(z).X(z)$$

# Passing a random process through an LTI system


Let, x(t) be a random process which its average varies by time.




Therefore, y(t) is a random process which its average varies by time.




$$y(t)=h(t)*x(t)=\int_{-\infty}^{+\infty} x(t-\tau).h(\tau)\,d\tau=\int_{-\infty}^{+\infty} h(t-\tau).x(\tau)\,d\tau$$

# Passing a random process through an LTI system

Let, x(t) be a random process which:

$$\mu_x(t),\ R_{xx}(t) \quad \text{are known.}$$

In LTI system we know:

$$y(t)=h(t)*x(t)=\int_{-\infty}^{+\infty} h(t-\tau).x(\tau)\,d\tau$$

**And now we want to know:** $\mu_y(t),\ R_{xy}(t),\ R_{yy}(t)\ ??$

$\mu_y(t)=??$ ………… $\Rightarrow$ $\mu_y(t)=\mu_x(t)*h(t)$

$R_{xy}(\tau)=??$ ………… $\Rightarrow$ $R_{xy}(\tau)=R_{xx}(\tau)*h(\tau)$

$R_{yy}(\tau)=??$ ………… $\Rightarrow$ $R_{yy}(\tau)=R_{xx}(\tau)*h(\tau)*h(-\tau)$

**These are important formula**

# Passing a random process through an LTI system

By using Laplace transform and Z transform, we have:

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$\begin{cases} R_{xy}(s) = R_{xx}(s) H(s) \\ R_{xy}(z) = R_{xx}(z) H(z) \end{cases}$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

$$\begin{cases} R_{yy}(s) = R_{xx}(s) H(s) h(-s) \\ R_{yy}(z) = R_{xx}(z) H(z) H\left(z^{-1}\right) \end{cases}$$

In system identification:

**Input and output signals are known.** $\longrightarrow R_{yy}, R_{xy}, R_{xx} \longrightarrow \boxed{h(t)}$

# System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include:*

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

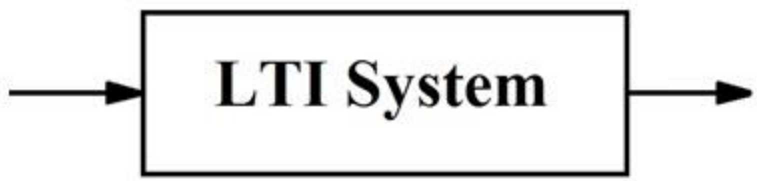
# System Identification by using correlation function

In system identification:


**Input and output signals are known.** $\longrightarrow R_{yy}, R_{xy}, R_{xx} \longrightarrow h(t)$

If we assume white noise input with unit variance then:

$R_{xx}(k)=\delta(k)$




$R_{xy}(k)=h(k)$

$$R_{xy}(\tau)=R_{xx}(\tau)*h(\tau)=h(\tau)$$

With this assumption that, the random process is ergodic.

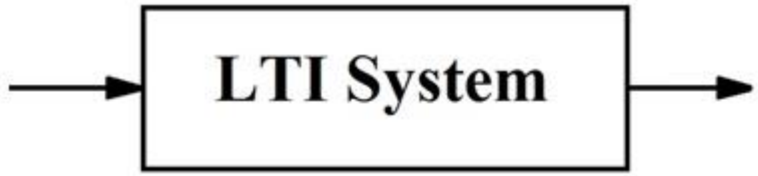
$$h(\tau)=R_{xy}(\tau)=E(x_i.y_{i+\tau})=\frac{1}{N}\sum_{i=1}^{N}x_i.y_{i+\tau}$$

The value of x(t)?? (Is it available?)

# System Identification by using correlation function

If we assume white noise input with unit variance then:

$$R_{xx}(k)=\delta(k)$$




$$R_{xy}(k)=h(k)$$

$$R_{xy}(\tau)=R_{xx}(\tau)*h(\tau)=h(\tau)$$

$$h(\tau)=R_{xy}(\tau)=E(x_i\,.y_{i+\tau})=\frac{1}{N}\sum_{i=1}^{N}x_i\,.y_{i+\tau}$$

If x(t) is not available then:

$$R_{yy}(\tau)=R_{xx}(\tau)*h(\tau)*h(-\tau)=h(\tau)*h(-\tau)$$

$$R_{yy}(s)\rightarrow H(s).H(-s) \qquad R_{yy}(z)\rightarrow H(z).H(z^{-1})$$

How to calculate H(z)/H(s)?

# System Identification by using correlation function

## Identification procedure by using correlation function (if x is available)

1- Sample inputs and outputs.

$$\begin{cases} x_t : x_1, x_2, \ldots, x_N \\ y_t : y_1, y_2, \ldots, y_N \end{cases}$$

2- Calculate correlation functions.

$$R_{xx}(\tau) = \frac{1}{N} \sum_{k=1}^{N} x_k \, x_{k+\tau} \qquad \tau = 0,1,2,\ldots$$

$$R_{xy}(\tau) = \frac{1}{N} \sum_{k=1}^{N} x_k \, y_{k+\tau} \qquad \tau = 0,1,2,\ldots$$

Mention: in the cases $k+\tau > N$, *refer to the beginning of numbers.*

3- We get Z transform from the result functions. $R_{xx}(z)$ , $R_{xy}(z)$

4- Calculate H(z) in the following equation.

$$H(z) = \frac{R_{xy}(z)}{R_{xx}(z)} \qquad \text{Or} \qquad H(z) = \frac{y(z)}{x(z)}$$

# System Identification by using correlation function

## Identification procedure by using correlation function (if x is not available)

1- Sample outputs. $y_t : y_1, y_2, \ldots, y_N$

2- Calculate correlation functions.

$$R_{yy}(\tau) = \frac{1}{N} \sum_{k=1}^{N} y_k \, y_{k+\tau} \qquad \tau = 0, 1, -1, 2, -2, \ldots$$

Mention: in the cases $k+\tau > N$ , *refer to the beginning of numbers.*

3- We get Z transform from the result functions. $R_{yy}(z)$

4- Calculate H(z) in the following equation.

$$R_{yy}(z) = H(z).H(z^{-1})$$

# System Identification by using correlation function

**Example 4-1:**

$$x_t = 1, 0, 0, \ldots\ldots \longrightarrow \boxed{h(t) = \left(\frac{1}{2}\right)^t} \longrightarrow y_t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots, \left(\frac{1}{2}\right)^{N-1}$$

$$R_{xx}(0) = \frac{1}{N} \qquad R_{xx}(1) = 0 \qquad R_{xy}(0) = \frac{1}{N} \qquad R_{xy}(1) = \frac{1}{2N}$$

$$R_{xx}(2) = 0 \ \cdots \ R_{xx}(N-1) = 0 \qquad R_{xy}(2) = \frac{1}{2^2 N} \ \cdots \ R_{xy}(N-1) = \frac{1}{2^{N-1} N}$$

$$\Rightarrow R_{xx}(z) = \frac{1}{N}$$

$$\Rightarrow R_{xy}(z) = \frac{1}{N}\left[1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \ \ldots \ + \frac{1}{2^{N-1}}z^{-N+1}\right] \cong \frac{1}{N} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H(z) = \frac{R_{xy}(z)}{R_{xx}(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

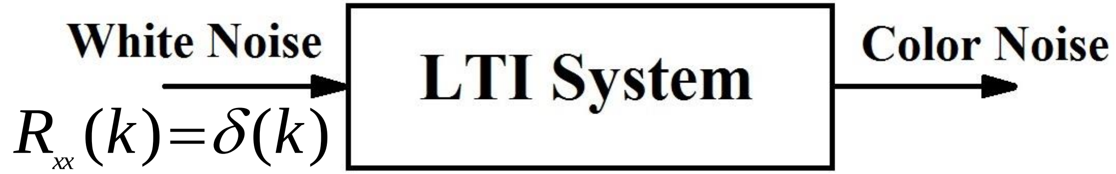
$$h(t) = \left(\frac{1}{2}\right)^t$$

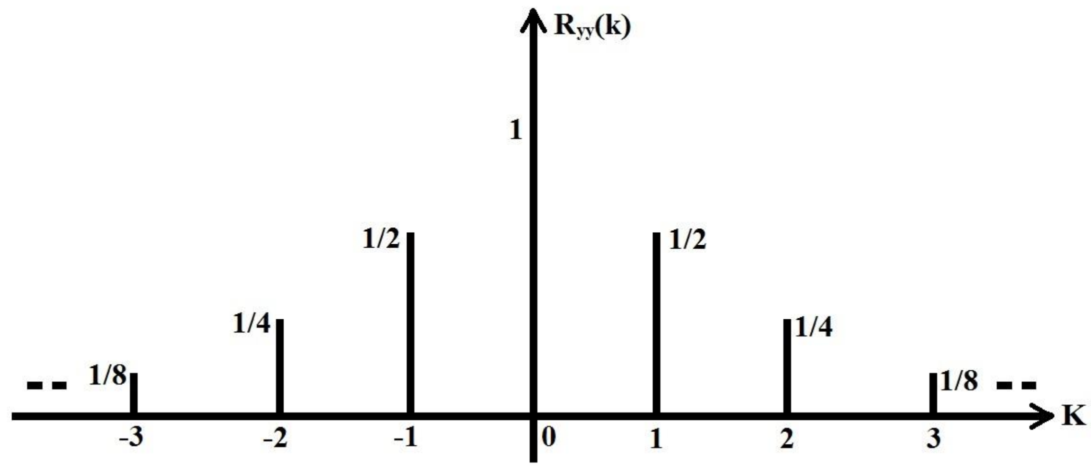


**Dr.** Ali Karimpour Nov 2015

# System Identification by using correlation function

## Example 4-2:




If we obtain Z transform from the signal:

$$R_{yy}(z)=\ldots+\frac{1}{8}z^{-3}+\frac{1}{4}z^{-2}+\frac{1}{2}z^{-1}+1+\frac{1}{2}z+\frac{1}{4}z^{2}+\frac{1}{8}z^{3}+\ldots.$$

$$R_{yy}(z)=\frac{\frac{3}{2}z}{\left(z-\frac{1}{2}\right)(2-z)}=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}\times\frac{1}{\left(1-\frac{1}{2}z\right)}\times\frac{3}{4}\qquad H(z)=\frac{\sqrt{3}}{2}\,\frac{1}{1-2^{-1}z^{-1}}$$

# System Identification by using correlation function

**Example 4-3:** Try to find the model through correlation method.

**LTI System**

$$y(t)+0.7y(t-1)=2u(t-1)$$

a) Let u(t)= $\delta_d$(t) and N=500 y(t): 0, 2, -1.4, 0.98, -0.686, …….

$$R_{uu}(0)=1/N \qquad R_{uu}(1)=0 \qquad R_{uy}(0)=0 \qquad R_{uy}(1)=2/N$$

$$R_{uy}(2)=-1.4/N \qquad R_{uy}(3)=0.98/N$$

$$\cdots\Rightarrow R_{uu}(z)=\frac{1}{N}$$

$$\cdots\Rightarrow R_{uy}(z)=\frac{1}{N}(\frac{2}{z}-\frac{1.4}{z^2}+\frac{0.98}{z^2}\ldots)$$

$$H(z)=\frac{R_{uy}(z)}{R_{uu}(z)}=\frac{2}{z}-\frac{1.4}{z^2}+\frac{0.98}{z^3}+\ldots.$$

$$H(z)=\frac{2}{z+0.7}$$

# System Identification by using correlation function

**Example 4-3(Continue):** Try to find the model through correlation method.

**LTI System**

$$y(t)+0.7y(t-1)=2u(t-1)$$

b) Let u(t) as a unit variance white noise. $R_{uu}(k)=\delta(k)$

y(t): 0, -2.5539,1.9679, …….

$R_{uy}(0)=\text{-}0.1082$ $R_{uy}(1)=2.1557$ $R_{uy}(1)=\text{-}1.7020$

$$H(z)=\frac{R_{uy}(z)}{R_{uu}(z)}=-0.1082+\frac{2.1557}{z}-\frac{1.7020}{z^2}+\ldots. \qquad H(z)=?$$

What is the drawback of this procedure?

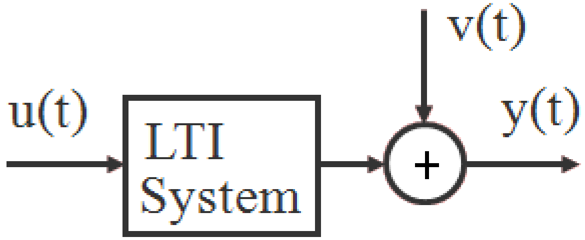
# System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include:*

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

# Disturbance models

There are always signals beyond our control. We assume that such effects can be lumped into an additive term *v*(*t*) at the output




**So**

$$y(t)=\sum_{k=1}^{\infty} g(k)\, u(t-k)+v(t)$$

There are many sources and causes for such a disturbance term.

- Measurement noise.
- Uncontrollable inputs. ( a person in a room produce 100 W/person)

# Disturbance models

## Disturbance **types**

1- Certain disturbance

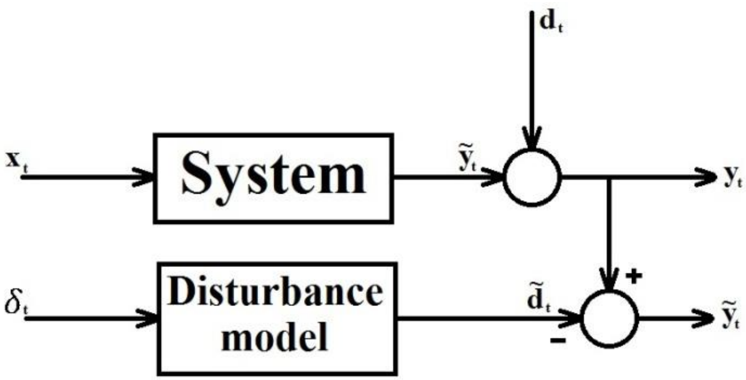
In each examination, disturbance is thoroughly similar and equal to the previous one.

2- Random disturbance

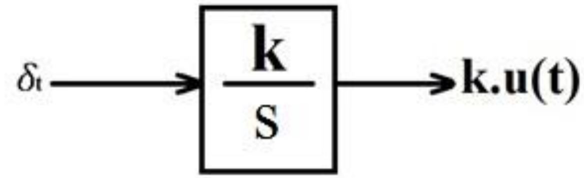
In each examination, it will be a different function of time.

# Disturbance models

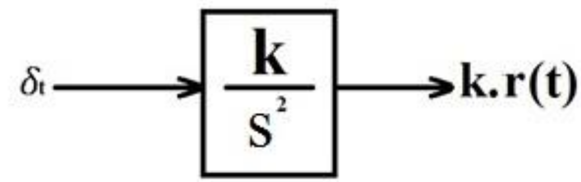
## Certain disturbance



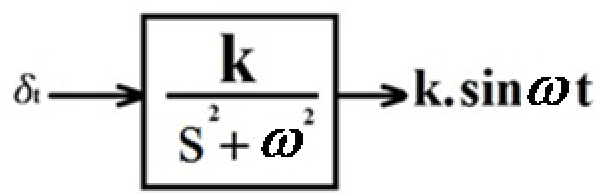

Bias model →




Drift model →




Sinusoid disturbance → model



# Disturbance models

## Characterization of disturbances

- Its value is not known beforehand.
- Making qualified guesses about future values is possible.
- It is natural to employ a probabilistic framework to describe future disturbances.

We put ourselves at time $t$ and would like to know disturbance at $t+k$, $k \geq 1$ so we use the following approach.

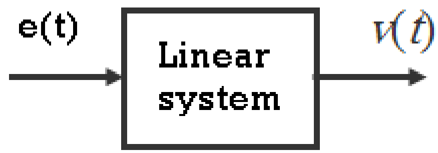
$$v(t) = \sum_{k=0}^{\infty} h(k)\ e(t-k)$$

Where $e(t)$ is a white noise.

This description does not allow completely general characteristic of all possible probabilistic disturbances, but it is versatile enough.

# Disturbance models

We will assume that *e*(*t*) is a white noise that has zero mean and variance λ .




Now we want to know the **characteristic of *v*(*t*)** :

$$v(t) = \sum_{k=0}^{\infty} h(k) e(t-k)$$

Mean:

$$Ev(t) = \sum_{k=0}^{\infty} h(k) Ee(t-k) = 0$$

Covariance:

$$Ev(t)v(t-\tau) =$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} h(k)h(s)Ee(t-k)e(t-\tau-s)$$

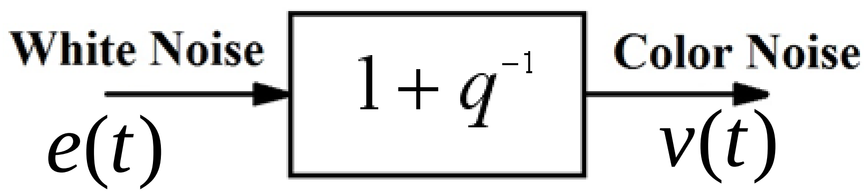
$$= \lambda \sum_{k=0}^{\infty} h(k)h(k-\tau) \quad = R_w(\tau)$$

**Since the mean and covariance are not depend on *t*, the process is said to be stationary.**

# Disturbance models

## Random disturbance Model

In general case, random process is assumed color.


$$R_{ee}(\tau)=\sigma_e^2\delta(\tau) \qquad v(t)=e(t)+e(t-1)$$

$$R_w(1)=E\{v(t)v(t+1)\}=E\{(e(t)+e(t-1))(e(t+1)+e(t))\}$$

$$R_w(1)=\sigma_e^2$$

So its a color noise.

# Disturbance models

## Parametric disturbance models


$$v(z) = H(z).e(z)$$

$$\frac{v(z)}{e(z)} = H(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_l z^{-l} = C(z^{-1})$$

$$v(k) = c_0 e(k) + c_1 e(k-1) + \ldots + c_l e(k-l)$$

**MA** (**M**oving **A**verage)

## MA model

Note: e(t) is white and it is a typical **time series.**

# Disturbance models

## Parametric disturbance models


$$v(z) = H(z).e(z)$$

$$\frac{v(z)}{e(z)} = H(z) = \frac{1}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \ldots + d_{nd} z^{-nd}} = \frac{1}{D(z^{-1})}$$

$$d_0 v(k) + d_1 v(k-1) + \ldots + d_{nd} v(k-n_d) = e(k)$$

AR (**A**uto **R**egressive)

## AR model

Note: e(t) is white and it is a typical **time series.**

# Disturbance models

## Parametric disturbance models


$$v(z) = H(z).e(z)$$

$$\frac{v(z)}{e(z)} = H(z) = \frac{c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_l z^{-l}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \ldots + d_{nd} z^{-nd}} = \frac{C(z^{-1})}{D(z^{-1})}$$

$$d_0 v(z) + d_1 z^{-1} v(z) + \ldots + d_{nd} z^{-nd} v(z) = c_0 e(z) + c_1 z^{-1} e(z) + \ldots + c_l z^{-l} e(z)$$

$$d_0 v(k) + d_1 v(k-1) + \ldots + d_{nd} v(k-n_d) = c_0 e(k) + c_1 e(k-1) + \ldots + c_l e(k-l)$$

AR MA

**ARMA model**

Note: e(t) is white and it is a typical **time series.**

# Disturbance models

AR ،MA ،ARMA Models (Time Series)

$$d_0 v(k) + d_1 v(k-1) + \ldots + d_{nd} v(k-n_d) = c_0 e(k) + c_1 e(k-1) + \ldots + c_l e(k-l)$$

AR MA

**ARMA model**

$C(z^{-1}) = 1$

$$d_0 v(k) + d_1 v(k-1) + \ldots + d_{nd} v(k-n_d) = e(k)$$

AR

**AR model**

$D(z^{-1}) = 1$

$$v(k) = c_0 e(k) + c_1 e(k-1) + \ldots + c_l e(k-l)$$

MA

**MA model**

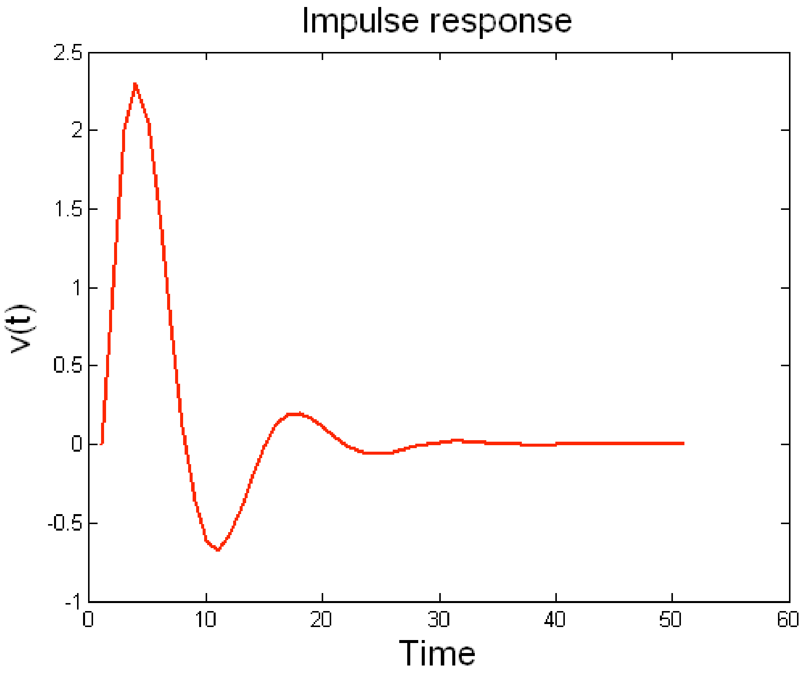
# Disturbance models

**Example 4-4:** Let v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1), derive impulse response h(k).

There are two methods to derive h(k).

In the first approach:

let e(0)=1, e(1)=0, e(2)=0, e(3)=0, …..




Clearly v(k)=h(k):

v(0)=h(0)=1,

v(1)=h(1)=1.5+0.5=2,

v(2)=h(2)=1.5*2-0.7=2.3,

…….

So h(k):1, 2, 2.3, 2.05, 1.465, …….

# Disturbance models

**Example 4-4(Continue):** Let v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1), derive impulse response h(k).

In the second approach:

$(1-1.5q^{-1}+0.7q^{-2})v(t)=(1+0.5q^{-1})e(t)$

$$v(t)=\frac{q^2+0.5q}{q^2-1.5q+0.7}e(t)=H(q)e(t)$$

$$=\left(1+2q^{-1}+2.3q^{-2}+2.05q^{-3}+1.465q^{-4}+...\right)e(t)$$

So h(k):1, 2, 2.3, 2.05, 1.465, …….

# Disturbance models

**Example 4-5:** Let v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1).

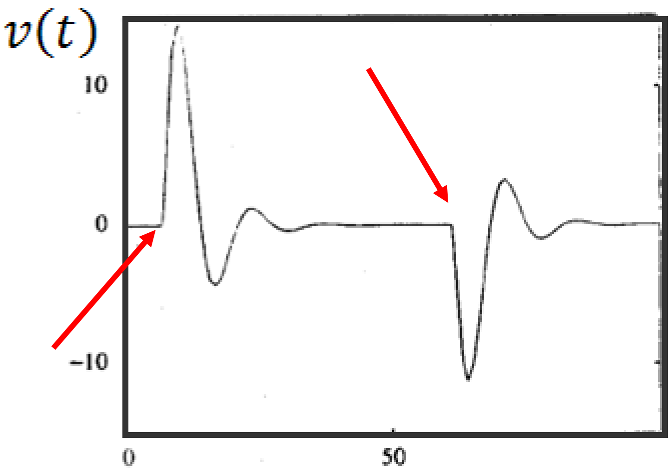
Consider for example, the following PDF for *e*(*t*):

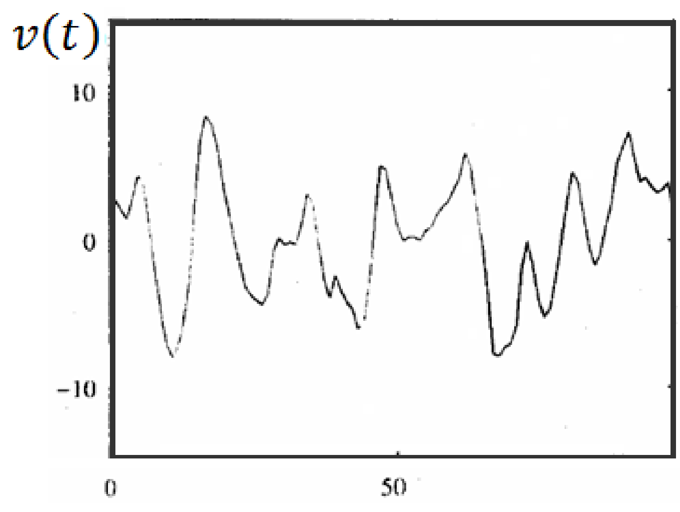
$$e(t) = 0, \quad with\ probability \quad 1-\mu$$

$$e(t) = r, \quad with\ probability \quad \mu$$

$$r \in N(0,\gamma)$$

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k)$$



# System Identification- A Non-parametric Approach (Correlation Method and Frequency Domain Method)

*Topics to be covered include:*

- Impulse responses, transfer functions.
- Passing a random process through an LTI system.
- System identification by using correlation function.
- Disturbance model.
- Frequency domain expression.

# Frequency-domain expressions

Let

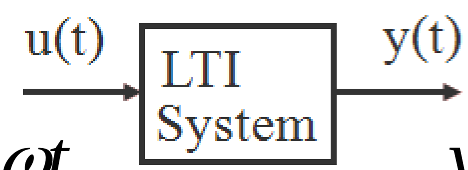
$$u(t)=\alpha\cos\omega t \qquad u(t)=\alpha\,\mathrm{Re}\left(e^{i\omega t}\right)$$


Now we can write output as:

$$y(t)=\alpha\sum_{k=1}^{\infty}g(k)\,\mathrm{Re}\left(e^{i\omega(t-k)}\right)=\alpha\,\mathrm{Re}\left(\sum_{k=1}^{\infty}g(k)e^{i\omega(t-k)}\right)$$

$$=\alpha\,\mathrm{Re}\left(e^{i\omega t}.\sum_{k=1}^{\infty}g(k)\left(e^{i\omega}\right)^{-k}\right)=\alpha\,\mathrm{Re}\left\{e^{i\omega t}G(e^{i\omega})\right\}$$

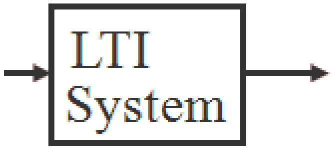
$$u(t)=\alpha\cos\omega t$$




$$y_{ss}(t)=\alpha\left|G(e^{i\omega})\right|\cos\left(\omega t+\angle G(e^{i\omega})\right)$$

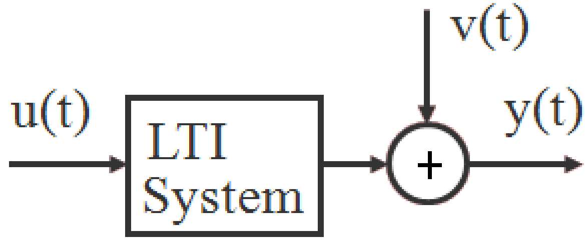
# Frequency-domain expressions

$u(t) = \alpha \cos \omega t$




$y_{ss}(t) = \alpha \left| G(e^{i\omega}) \right| \cos\left( \omega t + \angle G(e^{i\omega}) \right)$

Now let:




$$y_{ss}(t) = \alpha \left| G(e^{i\omega}) \right| \cos\left( \omega t + \angle G(e^{i\omega}) \right) + v(t)$$

**How to remove noise effect ????? Correlation method !!**

$$I_C(N) = \frac{1}{N} \sum_{t=1}^{N} y(t) \cos \omega t \qquad I_S(N) = \frac{1}{N} \sum_{t=1}^{N} y(t) \sin \omega t$$

$$I_C(N) \to \frac{\alpha}{2} \left| G(e^{j\omega}) \right| \cos \varphi \qquad I_S(N) = -\frac{\alpha}{2} \left| G(e^{j\omega}) \right| \sin \varphi$$

$$\Rightarrow \left| G(e^{i\omega}) \right|, \ \varphi$$

# Periodograms of signals over finite intervals

## Fourier transform (FT)

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

$f(t)$ → $g(\omega)$

$$f(t) = \int_{-\infty}^{+\infty} g(\omega)e^{i\omega t}d\omega$$

## Discrete Fourier transform (DFT)

$$U_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} u(t)e^{-i\omega t}$$

$u(1)$ $u(2)$ $u(3)$

$u(4)$ ...... $u(N)$

$u_1^N$

$U_N(\frac{2\pi}{N})$ $U_N(\frac{4\pi}{N})$ $U_N(\frac{6\pi}{N})$

$U_N(\frac{8\pi}{N})$ ...... $U_N(\frac{2N\pi}{N})$

$$u(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} U_N(2k\pi / N)e^{i2k\pi t / N}$$

# Periodograms of signals over finite intervals

**Some property of $U_N(\omega)$**

$$U_N(-\omega) = \overline{U_N(\omega)}$$

$$U_N(\omega + 2\pi) = U_N(\omega) \qquad \rightarrow u(t) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} U_N(2k\pi / N) e^{i2k\pi t/N}$$

**$U_N(\omega)$ tell us the weight that the frequency $\omega$ carries in the decomposition. So**

$$\left| U_N(\omega) \right|^2$$

**Is known as the periodogram of the signal u(t), t= 1, 2, 3, …..**

**Parseval's relationship:**

$$\sum_{k=1}^{N} \left| U_N(2k\pi / N) \right|^2 = \sum_{t=1}^{N} u(t)^2$$

# Periodograms of signals over finite intervals

**Example 4-6: Periodogram of a sinusoid**

$$u(t) = \alpha \cos \omega_0 t$$

$\omega_0 = 2\pi / N_0$ for some integer $N_0 > 1$

Let $t = 1, 2, 3, \ldots, N$ where $N$ is a multiple of $N_0$ $(N = sN_0)$

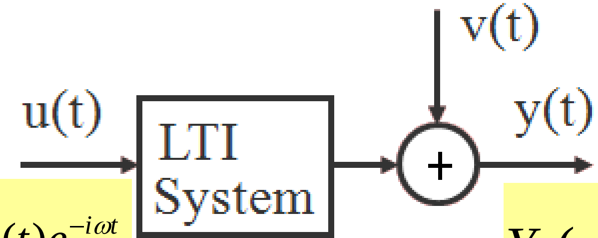
$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \alpha \cos \omega_0 t \, e^{-i\omega t} = \frac{\alpha}{2\sqrt{N}} \sum_{t=1}^{N} \left( e^{i\omega_0 t} + e^{-i\omega_0 t} \right) e^{-i\omega t}$$

$$= \frac{\alpha}{2\sqrt{N}} \sum_{t=1}^{N} \left( e^{i(\omega_0 - \omega)t} + e^{-i(\omega_0 + \omega)t} \right)$$

$$\left| U_N(\omega) \right|^2 = \begin{cases} N \dfrac{\alpha^2}{4} & if \ \ \omega = \pm \omega_0 \\ 0 & if \ \omega = \dfrac{2k\pi}{N}, \ k \neq s \end{cases}$$

# Frequency-domain expressions

Let {v(t)} is stationary stochastic process with spectrum $\Phi_v(\omega)$




$$U_N(\omega)=\frac{1}{\sqrt{N}}\sum_{t=1}^{N}u(t)e^{-i\omega t}$$

$$Y_N(\omega)=\frac{1}{\sqrt{N}}\sum_{t=1}^{N}y(t)e^{-i\omega t}$$

$$\hat{\hat{G}}_N(e^{i\omega})=\frac{Y_N(\omega)}{U_N(\omega)}$$

This is Empirical Transfer Function Estimation (ETFE) for arbitrary input.

Properties of the ETFE (Lemma 6.1 ref. [2])

- The ETFE is an asymptotically unbiased estimate.
- The variance of ETFE does not decrease as N increase. ( $\Phi_v(\omega)/|U_N(\omega)|^2$ )
- The estimate at different frequencies are asymptotically uncorrelated.

# Exercises

**Example 4-7:** Let y(t)-1.5y(t-1)+0.7y(t-2)=x(t-1)+0.5x(t-2). Sampling period is 0.1 Sec.

Let x(t) is a unit variance white noise (N=1000).

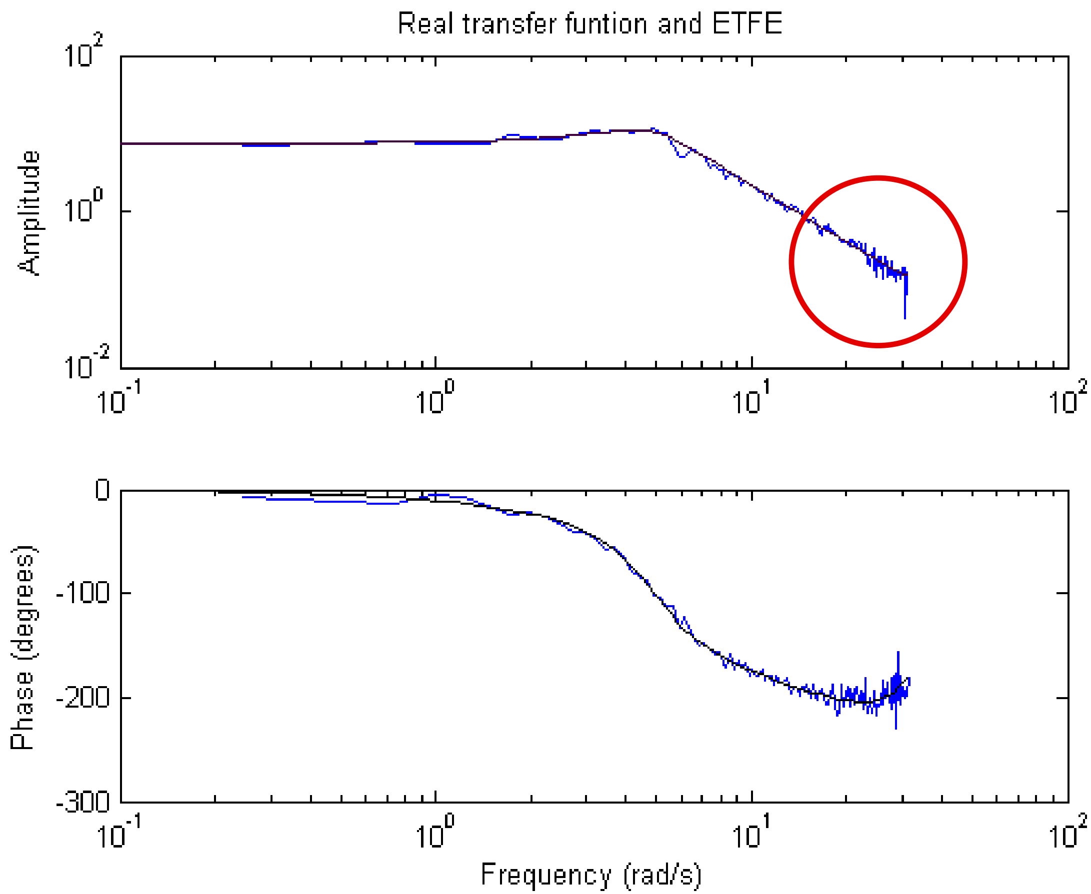
You can use etfe and bode m-file:

Note to frequency !!

Which property of ETFE leads to deviation !!

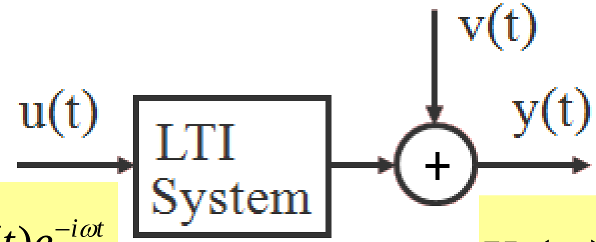
The variance of ETFE is given as the noise to signal ratio.

( $\Phi_v(\omega)/|U_N(\omega)|^2$ )



# Frequency-domain expressions

## Smoothed version of ETFE [2]




$$U_N(\omega)=\frac{1}{\sqrt{N}}\sum_{t=1}^{N}u(t)e^{-i\omega t}$$

$$Y_N(\omega)=\frac{1}{\sqrt{N}}\sum_{t=1}^{N}y(t)e^{-i\omega t}$$

$$\hat{\hat{G}}_N(e^{i\omega})=\frac{Y_N(\omega)}{U_N(\omega)}$$

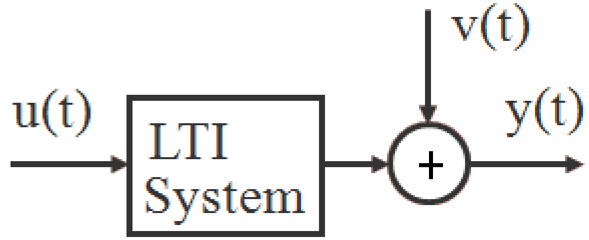
Properties of the ETFE (Lemma 6.1 ref. [2])

- The ETFE is an asymptotically unbiased estimate.
- The variance of ETFE does not decrease as N increase. ( $\Phi_v(\omega)/|U_N(\omega)|^2$ )
- The estimate at different frequencies are asymptotically uncorrelated.

“The true transfer function $G_0(e^{i\omega})$ is a smooth function of $\omega$ .”[2]

# Frequency-domain expressions

## Smoothed version of ETFE [2]




$$\hat{\hat{G}}_N(e^{i\omega}) = \frac{Y_N(\omega)}{U_N(\omega)}$$

$G_0(e^{i\omega})$ can be considered constant over an interval.

$$\frac{2\pi k_1}{N} = \omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega = \frac{2\pi k_2}{N}$$

$$\hat{G}_N(e^{i\omega_0}) = \frac{\sum_{k=k_1}^{k_2} \alpha_k \hat{\hat{G}}_N(e^{2\pi i k/N})}{\sum_{k=k_1}^{k_2} \alpha_k}$$

$$\alpha_k = \frac{|U_N(2\pi k/N)|^2}{\Phi_v(2\pi k/N)}$$

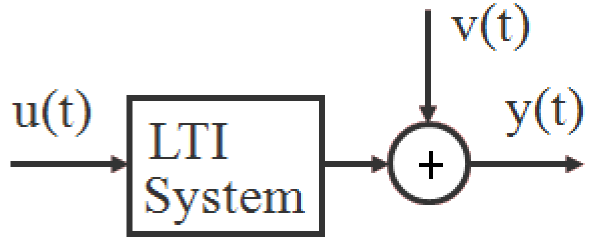
For large N:

$$\hat{G}_N(e^{i\omega_0}) = \frac{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)\hat{\hat{G}}_N(e^{i\xi})\,d\xi}{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)\,d\xi}.$$

$$\alpha(\xi) = \frac{|U_N(\xi)|^2}{\Phi_v(\xi)}$$

# Frequency-domain expressions

## Smoothed version of ETFE [2]




$$\hat{\hat{G}}_N(e^{i\omega}) = \frac{Y_N(\omega)}{U_N(\omega)}$$

$$\hat{G}_N(e^{i\omega_0}) = \frac{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)\hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{\xi=\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \alpha(\xi)d\xi}, \qquad \alpha(\xi) = \frac{|U_N(\xi)|^2}{\Phi_v(\xi)}$$

And more generally we use a window (W):

$$\hat{G}_N(e^{i\omega_0}) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_0)\alpha(\xi)\hat{\hat{G}}_N(e^{i\xi})d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_0)\alpha(\xi)d\xi}$$

Some windows for spectral analysis are:

Bartlett, Parzen and Hamming.

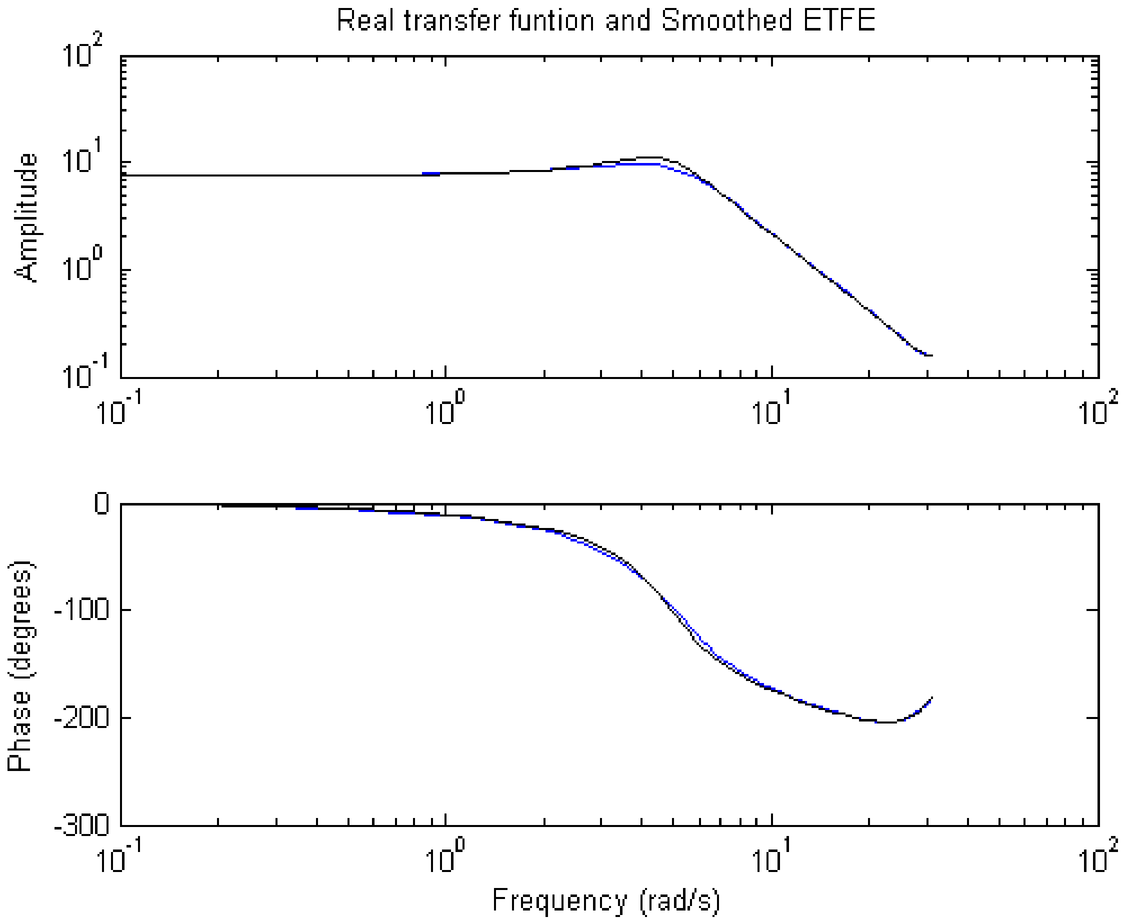
# Exercises

**Example 4-8:** Let y(t)-1.5y(t-1)+0.7y(t-2)=x(t-1)+0.5x(t-2).
Sampling period is 0.1 Sec.

Let x(t) is a unit variance white noise (N=1000).

You can use spa and bode m-file:

spa m-file use Hamming window.



# Signal Spectra

## A Common Framework for Deterministic and Stochastic Signals

In this course, we shall frequently work with signals that are described as *stochastic processes with deterministic components*.

$$y(t) = G(q)u(t) + H(q)e(t) \quad \Rightarrow \quad Ey(t) = G(q)u(t) \quad \Rightarrow$$

***y(t)* is not a stationary process**

To deal with this problem, we introduce the following definition:

**Quasi-stationary signals:**

A signal {s(t)} is said to be quasi-stationary if it is subject to

$$(i) \;\; Es(t) = m_s(t), \qquad |m_s(t)| \le C \quad \forall t$$

$$(ii) \;\; Es(t)s(r) = R_s(t,r), \qquad |R_s(t,r)| \le C$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} R_s(t, t-\tau) = R_s(\tau), \quad \forall \tau$$

$$\bar{E}s(t)s(t-\tau) = R_s(\tau)$$

# Definition of Spectra

When following limits exists:

$$\bar{E}s(t)s(t-\tau) = R_s(\tau)$$

We define the (power) ***spectrum*** of {s(t)} as

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\tau\omega}$$

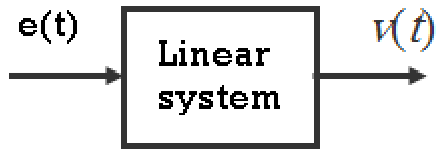
and ***cross spectrum*** between {s(t)} and {w(t)} as

$$\Phi_{sw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{sw}(\tau)e^{-i\tau\omega}$$

# Definition of Spectra

## Example of Stationary Stochastic Processes

**We will assume that *e*(*t*) is a white noise that has zero mean and variance λ .**




**Clearly v(t) is stationary**

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k)$$

$$R_e(\tau) = \lambda\delta(\tau) \qquad R_v(\tau) = Ev(t)v(t-\tau) = \lambda \sum_{k=\max(0,\tau)}^{\infty} h(k)h(k-\tau)$$

**The spectrums are:**

$$\Phi_e(\omega) = \sum_{\tau=-\infty}^{\infty} R_e(\tau)e^{-i\tau\omega} = \lambda$$

$$\Phi_v(\omega) = \sum_{\tau=-\infty}^{\infty} R_v(\tau)e^{-i\tau\omega} = \sum_{\tau=-\infty}^{\infty} \lambda \sum_{k=\max(0,\tau)}^{\infty} h(k)h(k-\tau)e^{-i\tau\omega} = \ldots\ldots = \lambda\left|H(e^{i\omega})\right|^2$$

**Where** 

$$H(e^{i\omega}) = \sum_{s=1}^{\infty} h(s)e^{-is\omega}$$

# Signal Spectra

## Spectrum of a Mixed Deterministic and Stochastic Signal

$$s(t) = u(t) + v(t)$$

deterministic

Stochastic: stationary and zero mean

$$\Phi_s(\omega) = \Phi_u(\omega) + \Phi_v(\omega)$$

# Transformation of Spectra by Linear Systems

**Theorem 4-1 :** Let{y(t)} be a given by

$$y(t) = G(q)u(t) + H(q)e(t)$$

where {u(t)} is a quasi-stationary, deterministic signal with $\Phi_u(\omega)$, and {e(t)} is white noise with variance λ. Let G and H be stable filters. Then {y(t)} is quasi-stationary and

$$\Phi_y(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \lambda \,|\, H(e^{i\omega}) \,|^2$$

$$\Phi_{yu}(\omega) = G(e^{i\omega})\Phi_u(\omega)$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

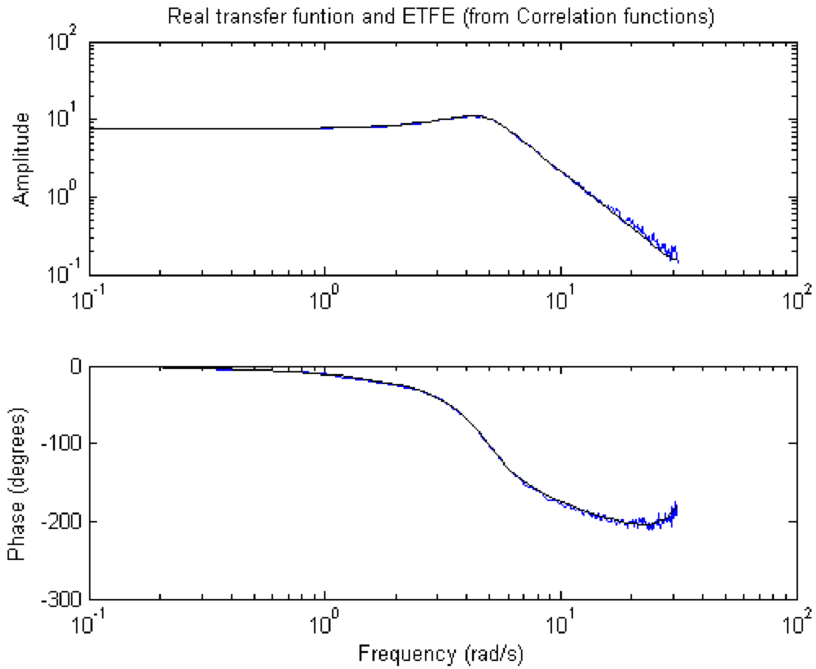
# Exercises

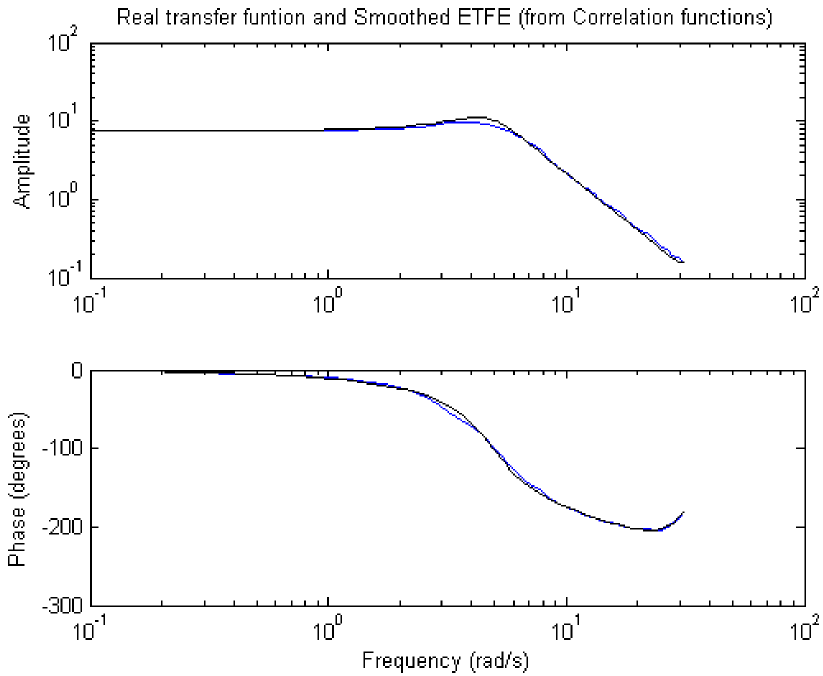
**Example 4-9:** Let y(t)-1.5y(t-1)+0.7y(t-2)=x(t-1)+0.5x(t-2).
Sampling period is 0.1 Sec. Let x(t) is a unit variance white noise (N=1000).

a) Derive $R_x(\tau)$ and $R_{xy}(\tau)$.

b) Derive ETFE through $R_x(\tau)$ and $R_{xy}(\tau)$.

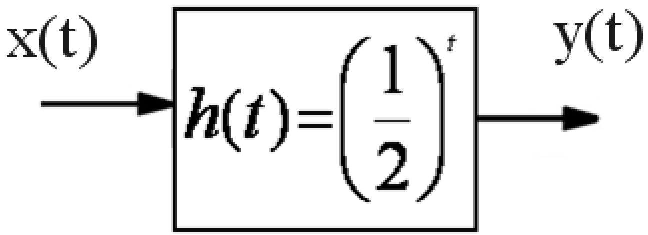
c) Derive Smoothed version of ETFE through $R_x(\tau)$ and $R_{xy}(\tau)$.



# Exercises

**Exercise 4-1:** Apply a unit step to following system(x(t):1, 1, 1, …..).




a) Derive output of system(y(t))

b) Derive $R_{xx}(\tau)$, $R_{xx}(z)$ and x(z).

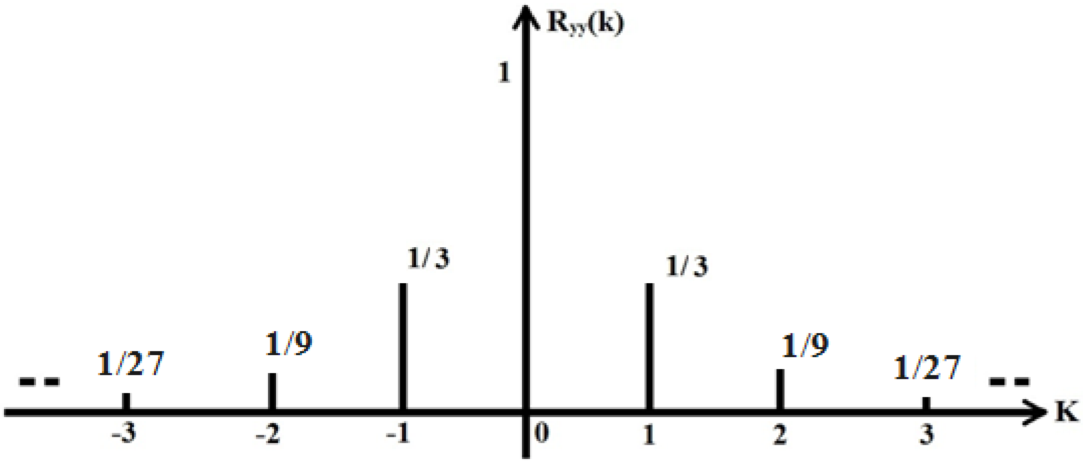
c) Derive $R_{xy}(\tau)$ and y(z).

d) Derive H(z).

# Exercises

**Exercise 4-2:** A unit variance white noise applied to following system


Suppose $R_{yy}(k)$ is in the following figure.



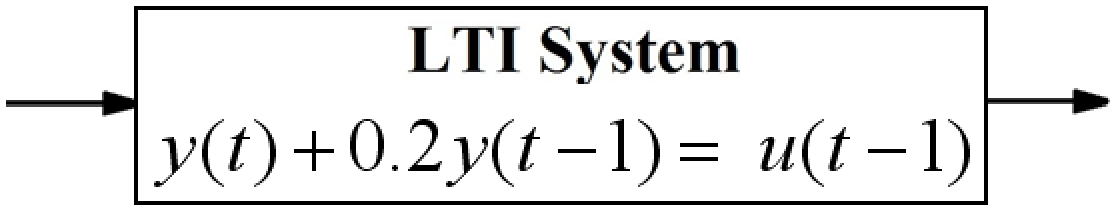

a) Derive $R_{yy}(z)$.

b) Derive H(z).


Dr. Ali Karimpour Nov 2015

# Exercises

**Exercise 4-3:** Try to find the model through correlation method.




a) Let u(t)=$\delta_d$(t) and N=500 derive y(t). Show y(t) for t=0:20.

b) By use of u(t) and y(t) derive H(z).

**Exercise 4-4:** Let v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1).

Consider the following PDF for $e(t)$ and derive v(t) for t=0:100.

$$e(t) = 0, \quad \text{with probability } 0.98$$

$$e(t) = r, \quad \text{with probability } 0.02 \quad r \in N(0, 2)$$

Plot v(t) for ten different examinations.

# Exercises

**Exercise 4-5:** Let v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1).

Consider the following PDF for $e(t)$ and derive v(t) for t=0:100.

$$e(t) = 0, \quad \text{with probability } 0.02$$

$$e(t) = r, \quad \text{with probability } 0.98 \quad r \in N(0,2)$$

Plot v(t) for ten different examinations.

**Exercise 4-6:** (Spectra of a Sinusoid) Consider

$$u(t) = A\cos\omega_0 t \qquad \text{to the interval} \qquad [1,\infty)$$

Show that

$$\Phi_u(\omega) = \frac{A^2}{4}\big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big).2\pi$$

Compare it with periodogram of a sinusoid(It is derived in the lecture).

# Simulation part

**Exercise4-7:** ARMA Processes: Let

$$v(t)-1.5v(t-1)+0.7v(t-2)=e(t)+0.5e(t-1)$$

Suppose that e(t) is:

$$e(t)=0, \quad \textit{with probability}\ 1-\mu$$

$$e(t)=r, \quad \textit{with probability}\ \mu \qquad r\in N(0,\gamma)$$

a) Derive $v(t)$ and plot it for μ=0.02 and $\gamma=2$, for t=0 till 100.

b) Derive $v(t)$ and plot it for μ=1 for t=0 till 100.

c) Derive the periodogram of the realization part b.

d) Derive the spectrum of the realization part b.

# Exercises

**Exercise4-8:** Let y(t)-1y(t-1)+0.5y(t-2)=x(t-1)+0.5x(t-2).
Sampling period is 0.2 Sec. Let x(t) is a unit variance white noise (N=1000).

a) Derive y(t) and show it for t=1:10.

b) Show real transfer function and ETFE in a figure.

c) Show real transfer function and smoothed ETFE in a figure.

d) Derive $R_x$ and $R_{xy}$ and show them for t=1:10.

e) Show real transfer function and ETFE(Derived from correlation data) in a figure.

f) Show real transfer function and smoothed ETFE(Derived from correlation data) in a figure.

# Exercises

**Exercise4-9:** Let y(t)-2y(t-1)+0.5y(t-2)=x(t-1)+0.5x(t-2).
Sampling period is 0.2 Sec. Let x(t) is a unit variance white noise (N=1000).

a) Derive y(t) and show it for t=1:10.

b) Show real transfer function and ETFE in a figure.

c) Is there any problem? Why?

# References


1- “System Modeling and Identification” Rolf Johansson (2010)

2- “System Identification Theory For The User” Lennart Ljung(1999)

3- “Perspectives on System Identification” Lennart Ljung (2009)

4- “Lessons in Digital Estimation Theory” Jerry M. Mendel (1987)

5- “Nonlinear System Identification” Oliver Nelles (2001)

6- “System Identification” M. Karrari (2012) (Farsi)